Probability and Statistics

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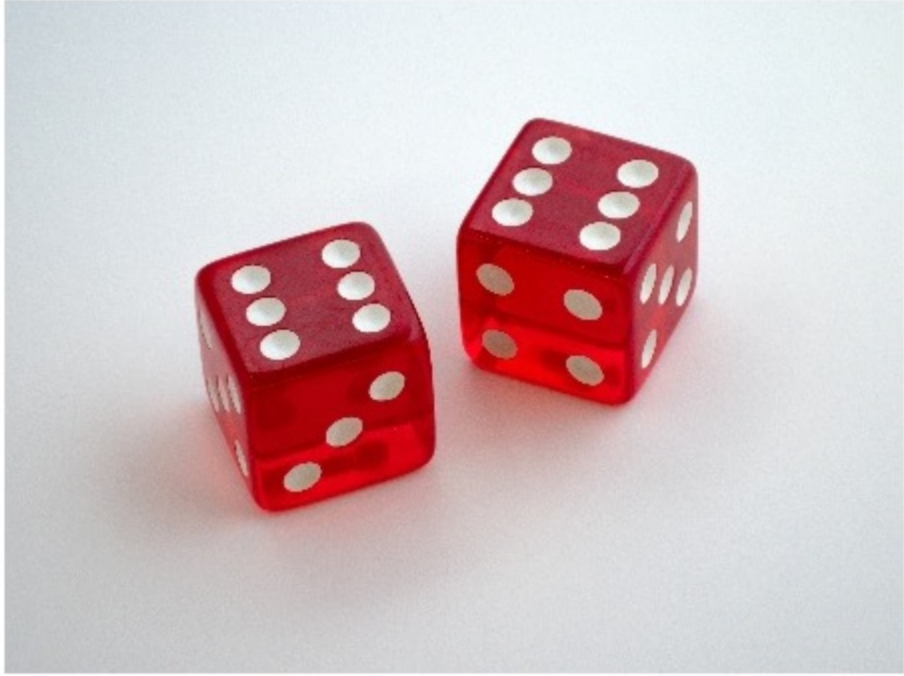

Rolling a Die Creates a Random Variable

Random Variable

X	Probability(X)
1	⅙
2	⅙
3	⅙
4	⅙
5	⅙
6	⅙

Die Rolls Are Uniform Probabilities

- If we roll a 6-sided die, what is the probability of rolling a 1?
- What is the probability of rolling an even number?

The Expected Value is the "Average" Roll

- When we roll a 6-sided die, what is the “most likely” value?
- Imagine rolling the die 100 times, what would the "average" roll be?
- The expected value of a random variable can be thought of as the *mean* or *average*

Relationships Among Random Variables

Independent variables: knowing one event has happened does not change the probability that the other happens

- Probability of rolling a 1 and flipping a head
- When X and Y are independent, $P(X \text{ and } Y) = P(X)P(Y)$

Dependent variables: knowing one event has happened gives us new information, affecting the probability that the other happens

- Probability that the sum of two die rolls being a 5, if the first roll was a 3

Conditional Probability

The probability of X given Y has occurred is P(X|Y), for example,

$$P(\text{sum} = 5 | \text{first die} = 3) = P(\text{sum is 5 if first die is 3}) = \frac{1}{6}$$

Conditional Probability

For random variables X and Y,

$$P(X|Y) = \frac{P(X \text{ and } Y)}{P(Y)}$$

Conditional Probability → $P(X|Y)$

Joint Probability → $P(X \text{ and } Y)$

Marginal Probability → $P(Y)$

Conditional Probability: Example

- By looking at a table of all possibilities, we found that

$$P(\text{sum} = 5|\text{first die} = 3) = \frac{1}{6}$$

- Now, using $P(X|Y) = \frac{P(X \text{ and } Y)}{P(Y)}$, we can calculate this without needing to find every possible value of two dice rolls:

$$\begin{aligned} P(\text{sum} = 5|\ \text{first die} = 3) &= \frac{P(\text{second die} = 2 \text{ and first die} = 3)}{P(\text{first die} = 3)} \\ &= \frac{1/36}{1/6} \\ &= \frac{1}{6} \end{aligned}$$

Conditional Probabilities Are Not Joint Probabilities

If we let X be the first die roll of value 3, and Y be the second of value 2, and Z be the sum, then

- Conditional probability:
  $P(Z|X) \longrightarrow P(\text{sum} = 5|\text{first die} = 3) = \frac{1}{6}$
- Joint probability:
  $P(X \text{ and } Y) \longrightarrow P(\text{roll 2 and 3}) = \frac{2}{36}$
- Probability of Z
  $P(Z) = P(X + Y) \longrightarrow P(\text{sum} = 5) = \frac{4}{36}$

Conditional, Joint, & Marginal Probabilities Are Related

Let X and Y be random variables.

1. If X and Y are independent then $P(X|Y) = \frac{P(X \text{ and } Y)}{P(Y)} = \frac{P(X)P(Y)}{P(Y)} = P(X)$

   Example: let X be the outcome of rolling a 6-sided die and let Y be the outcome of flipping a coin. Suppose we know that Y is "heads." What is the probability that we roll a 3? This tells us that $P(\textbf{roll 3}|\textbf{heads}) = P(\textbf{roll 3})$. *This matches intuition — flipping a coin does not change the outcome of rolling a die.*

Conditional, Joint, & Marginal Probabilities Are Related

Let X and Y be random variables.

2. $P(X) = \sum_{Y} P(X|Y)P(Y)$

   Example: let X be the sum of rolling two dice and let Y be the outcome of the first die roll. If we want to know the probability that X is 3, then this tells us $P(\text{sum is 3}) = \sum_{Y} P(\text{sum is 3}|\text{first roll was Y})P(\text{first roll was Y})$*. From here, we know that the sum can never be 3 unless the first roll is 1 or 2. Thus,*

   $$\begin{aligned} P(\text{sum is 3}) &= P(\text{sum is 3}|\text{first roll was 1})P(\text{first roll was 1}) + P(\text{sum is 3}|\text{first roll was 2})P(\text{first roll was 2}) \\ &= \frac{1}{6} \cdot \frac{1}{6} + \frac{1}{6} \cdot \frac{1}{6} = \frac{2}{36} \end{aligned}$$

Conditional, Joint, & Marginal Probabilities Are Related

Let X and Y be random variables.

3. $P(X) = \sum_{Y} P(X \text{ and } Y)$

Example: let X be the outcome of a first die roll and let Y be the outcome of a second die roll. If we want to find the probability that X is 3, this tells us that

$$P(X = 3) = \sum_{y=1}^{6} P(X = 3 \text{ and } Y = y)$$
$$= \sum_{y=1}^{6} \frac{1}{36} = \frac{6}{36} = \frac{1}{6}$$

Conditional, Joint, & Marginal Probabilities Are Related

Let X and Y be random variables.

1. If X and Y are independent then $P(X|Y) = \frac{P(X \text{ and } Y)}{P(Y)} = \frac{P(X)P(Y)}{P(Y)} = P(X)$
2. $P(X) = \sum_{Y} P(X|Y)P(Y)$
3. $P(X) = \sum_{Y} P(X \text{ and } Y)$

Conditional Probability

Joint Probability

Marginal Probability

Conditional Probabilities and Bayes' Theorem

Sometimes we want to find $P(X|Y)$ when we already know $P(Y|X)$.

Bayes' Theorem

For random variables X and Y,

$$P(X|Y) = \frac{P(Y|X)P(X)}{P(Y)}$$

Bayes' Theorem: Example

Sometimes we want to find $P(X|Y)$ when we already know $P(Y|X)$.

For instance, $P(\text{first die} = 3|\text{sum} = 5) = \frac{1}{4}$.

We can verify this using Bayes' Theorem.

$$\begin{aligned} P(\text{first die} = 3|\text{sum} = 5) &= \frac{P(\text{sum} = 5|\text{first die} = 3)P(\text{first die} = 3)}{P(\text{sum} = 5)} \\ &= \frac{\frac{1}{6} \cdot \frac{1}{6}}{\frac{4}{36}} \\ &= \frac{1}{4} \end{aligned}$$

Sample Exercise: Peanut Chocolate Detector

Assumptions: Suppose we have a new device that distinguishes whether or not a type of chocolate contains peanuts. If a chocolate contains peanuts, 99% of the time it correctly reports a positive result. Likewise, if a chocolate does not contain peanuts, 99% of the time it correctly reports a negative result. Imagine 1% of all chocolates contain peanuts.

Question: If the device reports that a chocolate contains peanuts, what is the probability that the chocolate *actually does* contain peanuts?

Sample Exercise: Peanut Chocolate Detector

p = random variable indicating whether peanuts are in a chocolate bar

d = random variable indicating whether we detected peanuts in a chocolate bar.

We were given $P(p) = 0.01$, $P(d|p) = 0.99$, and $P(\text{not } d|\text{not } p) = 0.99$.

We can then calculate $P(d|\text{not } p) = 0.01$ and $P(\text{not } p) = 0.01$.

Then, $P(d) = P(d|p)P(p) + P(\text{not } d|\text{not } p)P(\text{not } p)$.

By Bayes' Theorem, $P(p|d) = \frac{P(d|p)P(p)}{P(d)} = \frac{0.99 \cdot 0.01}{0.0198} = 0.5$.